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Geometric Aspects of Knot Modification of B-spline Surfaces

Miklós Hoffmann, Imre Juhász

Institute of Mathematics and Computer Science, Károly Eszterházy College
P.O.Box 43. H-3300 Eger, Hungary
email: hofi@ektf.hu

Department of Descriptive Geometry, University of Miskolc
H-3515 Miskolc-Egyetemváros, Hungary
email: agtji@gold.uni-miskolc.hu

Abstract. In a recent publication we described the effect of knot modifications of B-spline curves. The aim of this paper is the generalization of these results for surfaces. Altering one or two knot values of a B-spline surface, the paths of the points of the surface are discussed first, among which special ruled surfaces can be found. Then we prove that the family of B-spline surfaces obtained by knot alteration, possesses an envelope which is a lower order B-spline surface.

Key Words: B-spline surfaces, knot modification

MSC 2000: 53A05, 68U05

1. Introduction

B-spline and rational B-spline surfaces are popular and successful methods in Computer Aided Geometric Design. Beyond creating these types of surfaces the modification of existing ones are also of great importance. Based on their fairly simple data structure there are three ways to modify their shape: by repositioning the control points, by changing the knot vector or, in the rational case, by altering the weights.

Beside the most basic tools which can be found, e.g. in [7], one can find numerous papers presenting different ways of surface modification by moving simply one control point or changing one weight [6], or by more sophisticated methods (see e.g. [1], [3], [5] and references therein). Surprisingly enough none of the above mentioned tools applies knot alteration. Hence the purpose of the present paper is to describe the geometric properties of knot modifications of B-spline surfaces, that are partly based on the earlier results of the authors published in [4]. These theoretical results may form a basis starting from which, one can develop effective shape modification tools, as it was in the case of curves (see [2]).

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After the most basic definitions we will consider the individual points of a B-spline surface, which move along special curves and surfaces if one or two knot values of the original surface are modified. We will prove that these paths are rational curves and surfaces the degree of which alters from 1 to the degree of the surface, among which ruled and doubly ruled surfaces can also be found.

In the next section one- and two-parameter families of B-spline surfaces will be discussed which are also obtained by the modification of one or two knot values. Applying the derivatives of the basis functions with respect to a knot (cf. [4, 8]), here we will prove that all of these families have envelopes, which are lower order B-spline surfaces. Conclusions and directions of further research close the paper.

Throughout the paper we assume that the modified knots are of multiplicity of one, and we use the following basic definitions and notations (for the sake of simplicity the parameter domains $[u_{k-1}, u_{n+1}]$ and $[v_{l-1}, v_{m+1}]$ will be denoted by U and V, respectively):

Definition 1 *The curve* $\mathbf{a}(u)$ *defined by*

$$\mathbf{a}(u) = \sum_{r=0}^{n} N_r^k(u)\mathbf{d}_r\,, \quad u \in U$$

is called B-spline curve of order $1 < k \leq n+1$ *(degree* $k-1$*), where* $N_r^k(u)$ *is the* r^{th} *normalized B-spline basis function of order* k*, for the evaluation of which the knots* $u_0, u_1, \ldots, u_{n+k}$ *are necessary. Points* $\mathbf{d}_l$ *are called control points, while the polygon formed by these points is called control polygon.*

Definition 2 *The surface* $\mathbf{s}(u,v)$ *defined by*

$$\mathbf{s}(u,v) = \sum_{r=0}^{n}\sum_{s=0}^{m} N_r^k(u)N_s^l(v)\mathbf{d}_{rs}\,, \quad u \in U,\ v \in V \tag{1}$$

is called B-spline surface of order (k,l) *(degree* $(k-1,\, l-1)$*),* $(1 < k \leq n+1,\ 1 < l \leq m+1)$*, where* $N_r^k(u)$ *and* $N_s^l(v)$ *are the* r^{th} *and* s^{th} *normalized B-spline basis functions, for the evaluation of which the knots* $u_0, u_1, \ldots, u_{n+k}$ *and* $v_0, v_1, \ldots, v_{m+l}$ *are necessary, respectively. The points* $\mathbf{d}_{rs}$ *are called control points, while the mesh formed by these points is called control mesh.*

A *patch* of the B-spline surface can be written as

$$\mathbf{s}_{i,j}(u,v) = \sum_{r=i-k+1}^{i}\ \sum_{s=j-l+1}^{j} N_r^k(u)N_s^l(v)\mathbf{d}_{rs}, \quad u \in [u_i, u_{i+1}),\ v \in [v_j, v_{j+1}).$$

The u and v isoparametric curves of surface (1) are B-spline curves of order k and l, respectively. Fixing the parameter value $\tilde{u} \in U$ we obtain the isoparametric curve

$$\mathbf{b}(v) = \sum_{s=0}^{m} N_s^l(v)\mathbf{b}_s\left(\tilde{u}\right), \quad v \in V \tag{2}$$

the control points of which are

$$\mathbf{b}_s\left(\tilde{u}\right) = \sum_{r=0}^{n} N_r^k\left(\tilde{u}\right)\mathbf{d}_{rs} \tag{3}$$

The u isoparametric curves of the surface can be described in an analogous way.

2. Paths and path-surfaces

When a knot value u_p is altered, the shape of each u isoparametric curve of the surface changes on the range $[u_{p-k+1}, u_{p+k-1}] \cap U$, i.e., a strip of the surface is modified (cf. Fig. 2). Thus the alteration of the knot u_p effects at most $2\,(k-1)\,(m-l+2)$ patches on the rectangular domain

$$D_1 = ([u_{p-k+1}, u_{p+k-1}] \cap U) \times V.$$

The effect of the alteration of a knot v_q on the shape of the surface can analogously be described, where the domain of influence is

$$D_2 = ([v_{q-l+1}, v_{q+l-1}] \cap V) \times U.$$

If the knots u_p and v_q are simultaneously altered then: u_p effects the surface on the domain D_1, v_q effects on D_2, therefore the surface patches on the domain $D_1 \cup D_2$ are modified but both knots influence only patches on the domain $D_1 \cap D_2$ which results the modification of at most $4\,(k-1)\,(l-1)$ patches.

The alteration of the knot u_p modifies surfaces patches on the domain D_1 and individual points of the surface move along curves which we call *paths*, as we did in [4]. At first we prove, that these paths are rational curves the degree of which decreases symmetrically from k-1 to 1 as the indices of the surface patches getting farther from p. This fact immediately follows from the theorems proved for curves in [4].

Theorem 1 *Modifying the knot u_p the paths of points of surface patches*

$$\mathbf{s}_{p-g-1,h}(u,v) \text{ and } \mathbf{s}_{p+g,h}(u,v), \quad g = 0,1,\ldots,k-2;\ h = l-1, l, \ldots, m;\ (u,v) \in D_1$$

are rational curves of order $k-g$.

Proof. Let us consider the point corresponding to $(u,v) \in [u_i, u_{i+1}) \times [v_j, v_{j+1}) \subset D_1$. Fixing a v parameter value, we obtain the isoparametric curve

$$\mathbf{c}(u) = \sum_{r=0}^{n} N_r^k(u)\mathbf{c}_r(v)$$

the control points of which are

$$\mathbf{c}_r(v) = \sum_{s=0}^{m} N_s^l(v)\,\mathbf{d}_{rs}$$

Theorems 1 and 2 of [4] holds for the curve $\mathbf{c}(u)$ which completes the proof. □

Corollary 1 *Paths of the points of the surface patches $\mathbf{s}_{p-k+1,h}(u,v)$ and $\mathbf{s}_{p+k-2,h}(u,v)$, $(h = l-1, l, \ldots, m)$ are straight line segments that are parallel to the directions determined by the pair of points $\mathbf{c}_{p-k}(v), \mathbf{c}_{p-k+1}(v)$ and $\mathbf{c}_{p-1}(v), \mathbf{c}_p(v)$, respectively.*

Analogous results can be obtained for the points of the surface on D_2 by the modification of a v_q knot.

If the knots u_p and v_q are simultaneously altered, their joint effect modifies the patches on the domain $D_1 \cap D_2$. Points of these patches move on surfaces depending on the parameters u_p and v_q, which we will refer to as *path–surfaces*. At first we prove a general theorem for them, which states, that these surfaces are rational surfaces the degree of which decreases symmetrically in both direction. Then those ruled path-surfaces will be discussed that are obtained at the borders and the corners of the affected domain.

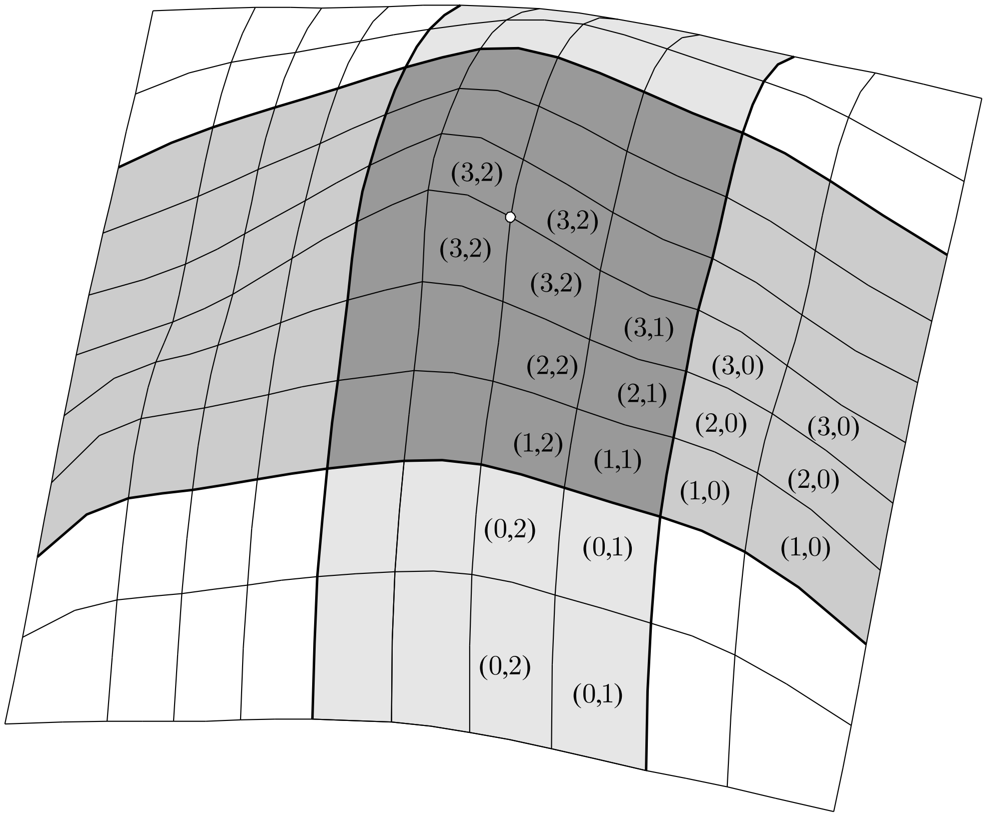

Figure 1: Modifying two knots of a (3,2) degree B-spline surface the individual points of patches move on rational curves and surfaces, the degrees of which are denoted by (n, m), where $(n, 0)$ and $(0, m)$ are for curves of degree n and m in the u and v direction, separately. The degrees decrease formally in a central symmetric way, the centre of which is denoted by the small circle.

Theorem 2 *If the knots u_p and v_q are simultaneously modified, points of the surface patches*

$$\begin{array}{ll} \mathbf{s}_{p-g-1,q-h-1}(u,v), & \mathbf{s}_{p-g-1,q+h}(u,v), \\ \mathbf{s}_{p+g,q-h-1}(u,v), & \mathbf{s}_{p+g,q+h}(u,v), \end{array} \qquad g = 0, 1, \ldots, k-2; \; h = 0, 1, \ldots, l-2$$

move on rational surfaces in u_p and v_q of order $(k-g, \; l-h)$.

Proof. Let us consider the point $(u, v) \in D_1 \cap D_2$ on the surface $\mathbf{s}(u, v)$ and its path-surface $\mathbf{s}(u, v, u_p, v_q)$. Fixing a parameter value $\tilde{v}_q$, we obtain a u_p isoparametric curve of the path-surface which is a rational curve of order $k - g$ due to Theorem 1. Analogously, for any fixed $\tilde{u}_p$ value, the obtained v_q isoparametric curve is a rational curve of order $l - h$. Thus the isoparametric curves of the path-surface are rational curves of degree $k - g$ and $l - h$, respectively, which completes the proof. □

Corollary 2 *Path-surfaces of the points at the borders of the array of the concerned patches are ruled surfaces, i.e., path-surfaces of points of the patches*

$$\begin{array}{ll} \mathbf{s}_{p-k+1,q-h-1}(u,v), & \mathbf{s}_{p-k+1,q+h}(u,v), \\ \mathbf{s}_{p+k-2,q-h-1}(u,v), & \mathbf{s}_{p+k-2,q+h}(u,v), \end{array} \qquad h = 0, 1, \ldots, l-2$$

and

$$\begin{array}{ll} \mathbf{s}_{p-g-1,q-l+1}(u,v), & \mathbf{s}_{p+g,q-l+1}(u,v), \\ \mathbf{s}_{p-g-1,q+l-2}(u,v), & \mathbf{s}_{p+g,q+l-2}(u,v), \end{array} \qquad g = 0, 1, \ldots, k-2$$

are ruled surfaces, but these are not cylinders in general, since the direction of their generators varies point by point.

Corollary 3 *Path-surfaces of the points at the four corner elements of the array of the concerned patches*

$$\begin{array}{ll} \mathbf{s}_{p-k+1,q-l+1}(u,v), & \mathbf{s}_{p-k+1,q+l-2}(u,v), \\ \mathbf{s}_{p+k-2,q-l+1}(u,v), & \mathbf{s}_{p+k-2,q+l-2}(u,v) \end{array}$$

are doubly ruled surfaces, i.e., both isoparametric curves of them are straight lines. Moreover, these doubly ruled surfaces are affine transforms of the surfaces obtained by the bilinear combination of the control point quadruples

$$\begin{array}{llll} \mathbf{d}_{p-k,q-l}, & \mathbf{d}_{p-k,q-l+1}, & \mathbf{d}_{p-k+1,q-l}, & \mathbf{d}_{p-k+1,q-l+1}; \\ \mathbf{d}_{p-k,q-1}, & \mathbf{d}_{p-k,q}, & \mathbf{d}_{p-k+1,q-1}, & \mathbf{d}_{p-k+1,q}; \\ \mathbf{d}_{p-1,q-l}, & \mathbf{d}_{p-1,q-l+1}, & \mathbf{d}_{p,q-l}, & \mathbf{d}_{p,q-l+1}; \\ \mathbf{d}_{p-1,q-1}, & \mathbf{d}_{p-1,q}, & \mathbf{d}_{p,q-1}, & \mathbf{d}_{p,q}; \end{array}$$

respectively. Such a bilinear blend is a hyperbolic paraboloid (saddle) in general, but in the case of coplanar control points it is a planar quadrilateral region.

Proof. We prove the statement for points of the patch $\mathbf{s}_{p-k+1,q-l+1}(u,v)$. The path-surface of the fixed point $(u,v) \in [u_{p-k+1}u_{p-k+2}) \times [v_{q-l+1}, v_{q-l+2})$ is

$$\begin{aligned} \mathbf{p}(u_p, v_q) &= \sum_{r=p-2(k-1)}^{p-k+1} \sum_{s=q-2(l-1)}^{q-l+1} N_r^k(u,u_p) N_s^l(v,v_q) \mathbf{d}_{rs} \\ &= \mathbf{C}_0 + \sum_{r=p-k}^{p-k+1} \sum_{s=q-l}^{q-l+1} N_r^k(u,u_p) N_s^l(v,v_q) \mathbf{d}_{rs} \end{aligned} \qquad u_p \in [u_{p-1}, u_{p+1}],\ v_q \in [v_{q-1}, v_{q+1}]$$

where

$$\mathbf{C}_0 = \sum_{r=p-2(k-1)}^{p-k-1} \sum_{s=q-2(l-1)}^{q-l-1} N_r^k\left(u,u_p\right) N_s^l\left(v,v_q\right) \mathbf{d}_{rs}$$

is independent of u_p and v_q, because neither u_p nor v_q appear in the coefficient functions. Basis functions appearing in the u_p and v_q dependent terms can be written in the form

$$\begin{aligned} N_{p-k}^k\left(u,u_p\right) &= C_1 + C_2\left(1 - \frac{u-u_{p-k+1}}{u_p-u_{p-k+1}}\right) \\ N_{p-k+1}^k\left(u,u_p\right) &= C_2\frac{u-u_{p-k+1}}{u_p-u_{p-k+1}} \\ N_{q-l}^l\left(v,v_q\right) &= C_3 + C_4\left(1 - \frac{v-v_{q-l+1}}{v_q-v_{q-l+1}}\right) \\ N_{q-l+1}^l\left(v,v_q\right) &= C_4\frac{v-v_{q-l+1}}{v_q-v_{q-l+1}} \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{u-u_{p-k}}{u_{p-1}-u_{p-k}} N_{p-k}^{k-1}(u), \qquad C_2 = \frac{u-u_{p-k+1}}{u_{p-1}-u_{p-k+1}} N_{p-k+1}^{k-2}(u), \\ C_3 &= \frac{v-v_{q-l}}{v_{q-1}-v_{q-l}} N_{q-l}^{l-1}(v), \qquad C_4 = \frac{v-v_{q-l+1}}{v_{q-1}-v_{q-l+1}} N_{q-l+1}^{l-2}(v) \end{aligned}$$

do not depend on u_p and v_q. For the sake of brevity we introduce the notations

$$P = \frac{u - u_{p-k+1}}{u_p - u_{p-k+1}} \quad \text{and} \quad Q = \frac{v - v_{q-l+1}}{v_q - v_{q-l+1}}$$

using which we can describe the path-surface as

$$\begin{aligned} \mathbf{p}\left(u_p, v_q\right) \quad = \quad & \mathbf{C}_0 + C_1 C_3 \mathbf{d}_{p-k,q-l} + C_2 C_3 \left(P \mathbf{d}_{p-k+1,q-l} + (1-P)\, \mathbf{d}_{p-k,q-l}\right) + \\ & C_1 C_4 \left(Q \mathbf{d}_{p-k,q-l+1} + (1-Q)\, \mathbf{d}_{p-k,q-l}\right) + \\ & C_2 C_4 \left[\begin{array}{cc} P & (1-P) \end{array}\right] \left[\begin{array}{cc} \mathbf{d}_{p-k+1,q-l+1} & \mathbf{d}_{p-k+1,q-l} \\ \mathbf{d}_{p-k,q-l+1} & \mathbf{d}_{p-k,q-l} \end{array}\right] \left[\begin{array}{c} Q \\ (1-Q) \end{array}\right]. \end{aligned} \qquad (4)$$

The last term of the right hand side is a bilinear combination of four control points, therefore it describes either a hyperbolic paraboloid or a planar quadrilateral region. This surface is translated by the initial three terms of the sum, the second and fourth terms of which are linear functions in P and Q, respectively. Such a linear functional translation results an affine transformation, thus the path-surface is an affine transform of the surface (4). □

Summarizing the preceding results one can observe, that simultaneously modifying the knot values u_p and v_q a topologically quadrilateral part of the surface is effected containing at most $4(k-1)\,(l-1)$ number of patches around the patch $\mathbf{s}_{p,q}(u,v)$. The points of these patches move on rational surfaces the degree of which decrease in a central symmetrical way as we consider farther patches in both parameter directions (cf. Fig. 1). Along the sides of this array of patches path-surfaces are ruled surfaces, while at the four corners one can find bilinear path-surfaces.

3. Envelope of the family of surfaces

Modifying the knot u_p of surface (1), we obtain a one-parameter family of surfaces (see Fig. 2) with the family parameter u_p, which will be denoted by

$$\mathbf{s}(u, v, u_p), \quad u \in U,\ v \in V,\ u_p \in [u_{p-1}, u_{p+1}],$$

while modifying two knots, u_p and v_q at the same time, a two-parameter family of surfaces can be obtained with the family parameters u_p and v_q. This surface will be denoted by

$$\mathbf{s}(u, v, u_p, v_q), \quad u \in U,\ v \in V,\ u_p \in \left[u_{p-1}, u_{p+1}\right],\ v_q \in \left[v_{q-1}, v_{q+1}\right].$$

In this section we will prove, that these families of surfaces always possess an envelope, which is a lower order B-spline surface with the same control points and knot values (except the modified one) as the members of the families.

Theorem 3 *Altering a knot value u_p of a (k,l) order B-spline surface $\mathbf{s}(u,v)$, the one-parameter family of surfaces*

$$\mathbf{s}\left(u, v, u_p\right) = \sum_{r=0}^{n} \sum_{s=0}^{m} N_r^k\left(u, u_p\right) N_s^l(v) \mathbf{d}_{rs}\,, \quad u \in U,\ v \in V,\ u_p \in \left[u_{p-1}, u_{p+1}\right]$$

has an envelope, which is a $(k-1,l)$ order B-spline surface

$$\mathbf{e}(w, v) = \sum_{r=p-k+1}^{p-1} \sum_{s=0}^{m} N_r^{k-1}(w) N_s^l(v) \mathbf{d}_{rs}\,, \quad w \in [w_{p-1}, w_p],\ v \in V$$

where the new knot vector is defined as

$$w_j = \begin{cases} u_j, & \text{if } j < p \\ u_{j+1}, & \text{if } j \geq p \end{cases},$$

i.e., the p^{th} *knot is left out from the original knots* $\{u_j\}$.

Proof. We prove, that for every permissible values of u_p the surfaces $\mathbf{e}(w,v)$ and $\mathbf{s}(u,v,u_p)$ have a point on common at the parameter values $w = u = u_p$ and the tangents of the isoparametric curves of both surfaces are parallel in these points. More precisely we prove that the equations

$$\mathbf{e}(u_p,v) = \mathbf{s}(u_p,v,u_p), \tag{5}$$

$$\frac{\partial}{\partial w}\mathbf{e}(w,v)\Big|_{w=u_p} = \frac{k-2}{k-1}\frac{\partial}{\partial u}\mathbf{s}(u,v,u_p)\Big|_{u=u_p}, \tag{6}$$

$$\frac{\partial}{\partial v}\mathbf{e}(u_p,v) = \frac{\partial}{\partial v}\mathbf{s}(u_p,v,u_p) \tag{7}$$

are fulfilled $\forall u_p \in [u_{p-1}, u_{p+1}]$.

For every fixed value of v we obtain an u isoparametric curve of the surface $\mathbf{s}(u,v)$ which generates a one-parameter family of curves of order k when the knot u_p is altered and we also obtain an u isoparametric curve of $\mathbf{e}(u,v)$. The equalities (5) and (6) are obviously fulfilled for these curves due to Theorem 3 of [4].

On the bases of equations (2) and (3), the v isoparametric curve on the surface $\mathbf{s}(u,v,\tilde{u}_p)$, $\tilde{u}_p \in [u_{p-1}, u_{p+1}]$, that belongs to the fixed u parameter value $u = \tilde{u}_p$, is determined by the control points

$$\mathbf{b}_s(\tilde{u}_p) = \sum_{r=0}^{n} N_r^k(\tilde{u}_p)\,\mathbf{d}_{rs}.$$

Fixing the parameter value $w = \tilde{u}_p$ of the surface $\mathbf{e}(w,v)$, we obtain a v isoparametric curve on the surface which curve is determined by the control points

$$\mathbf{c}_s(\tilde{u}_p) = \sum_{r=p-k+1}^{p-1} N_r^{k-1}(\tilde{u}_p)\,\mathbf{d}_{rs}.$$

According to Theorem 3 of [4] $\mathbf{b}_s(\tilde{u}_p) = \mathbf{c}_s(\tilde{u}_p)$, therefore the isoparametric curves $\mathbf{e}(\tilde{u}_p, v)$ and $\mathbf{s}(\tilde{u}_p, v, \tilde{u}_p)$ are identical for all permissible $\tilde{u}_p$, i.e., the equality (7) also holds. □

Corollary 4 *The surface* $\mathbf{e}(w,v)$ *touches the elements of the one-parameter family of surfaces* $\mathbf{s}(u,v,u_p)$ *along their isoparametric curves* $\mathbf{s}(u_p,v,u_p)$.

Now, we show a similar property of the two-parameter family of surfaces obtained by the simultaneous alteration of the knots u_p and v_q.

Theorem 4 *Simultaneously altering the knots* u_p *and* v_q *of the* (k,l) *order B-spline surface* $\mathbf{s}(u,v)$ *the two-parameter family of surfaces*

$$\mathbf{s}(u,v,u_p,v_q) = \sum_{r=0}^{n}\sum_{s=0}^{m} N_r^k(u,u_p) N_s^l(v,v_q)\mathbf{d}_{rs},$$

$$u \in U,\ v \in V,\ u_p \in [u_{p-1}, u_{p+1}],\ v_q \in [v_{q-1}, v_{q+1}]$$

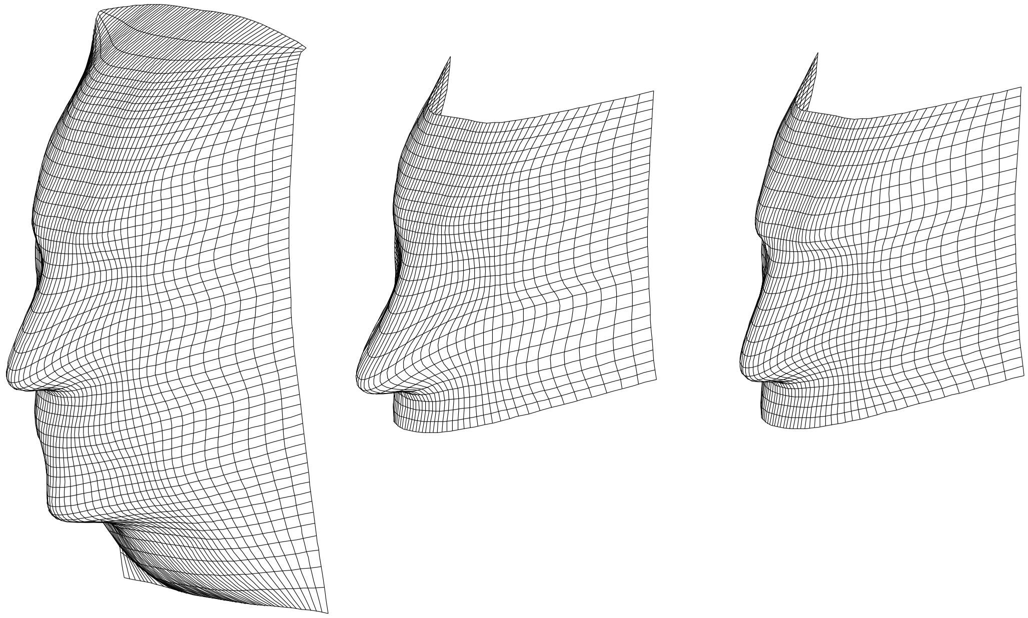

Figure 2: A (5,4) order B-spline surface and its modifications by altering a knot in the (vertical) u direction. Left: the original surface; middle: the modified strip of the surface if $u_i = u_{i-1}$ and right: if $u_i = u_{i+1}$.

has an envelope which is a $(k-1, l-1)$ *order B-spline surface*

$$\mathbf{e}(w,z) = \sum_{r=p-k+1}^{p-1} \sum_{s=q-l+1}^{q-1} N_r^{k-1}(w) N_s^{l-1}(z) \mathbf{d}_{rs}, \quad w \in [w_{p-1}, w_p],\ z \in [z_{q-1}, z_q]$$

where the new knot vectors are

$$w_j = \begin{cases} u_j, & \text{if } j < p \\ u_{j+1}, & \text{if } j \geq p \end{cases} \quad \text{and} \quad z_j = \begin{cases} v_j, & \text{if } j < q \\ v_{j+1}, & \text{if } j \geq q \end{cases},$$

i.e., the p^{th} *and the* q^{th} *knots are left out from the original knots* $\{u_j\}$ *and* $\{v_j\}$*, respectively.*

Proof. The consecutive application of Theorem 3 verifies the statement. □

4. Conclusion and further research

The geometrical aspects of knot modification of B-spline surfaces have been discussed in this paper. Based on our earlier results in terms of curves, we have proved that altering one or two knot values the points of the B-spline surfaces move along rational curves and surfaces, while the one- and two-parameter families of surfaces have an envelope, which is also a B-spline surface.

These theoretical result hopefully lead to practical shape control methods for B-spline surfaces as it was in the case of curves. In case of bicubic B-spline surfaces constrained shape control tools can be developed by a straightforward generalization of the methods described in [2], i.e., by altering three knot values simultaneously in both parameter directions. Extensions of the results for rational B-spline surfaces and for multiple knot values are also natural directions of our further research.

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